PERGAMON

International Journal of Solids and Structures 38 (2001) 8375–8388

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

An asymptotic theory for spherical shells

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Received 6 June 2000

Abstract

We derive the exact two-dimensional equations for a spherical shell from the three-dimensional elastic state by an asymptotic expansion. The resulting equations are presented as a power series in the thickness of the shell. Results, for which a three-dimensional solution is known or can be determined, show full agreement.

Keywords: Elasticity; Shell theory; Spherical shells; Asymptotic expansion

1. Introduction

The purpose of this paper is to present an exact theory for elastic spherical shells based on the three-dimensional theory of elasticity without any additional postulate like the "Kirchhoff hypothesis" or any conjecture regarding the elastic energy. The only assumptions made, are of a purely mathematical character concerning the differentiability and convergence of series. This paper may be considered as an extension of the asymptotic theory for cylindrical shells (Niordson, 2000). The lowest order bending theory of that asymptotic expansion gave more accurate results – in some cases considerably more accurate results – than results arrived at by commonly used engineering theories (Niordson and Niordson, 1999). There may therefore be reasons to expect a corresponding improvement for spherical shells.

The first attempts to solve the problem of the vibrations of a thin, elastic spherical shell precede the formulation of shell theories. Thus Lamb (1882) succeeded in adapting results derived by him earlier on the vibration of a spherical body to the case of an elastic body bounded by two concentric spherical surfaces. He also discussed the limit when the thickness approaches zero, i.e. a spherical membrane, which constitutes a *complete* sphere. Subsequent work on thin shells by Love (1888) leads to the classical theory of shells, which was applied to thin spherical shells by other writers (van der Neut, 1932; Havers, 1935; Federhofer, 1937a,b; Kalnins, 1963; Niordson, 1984). All these theories are of an approximate character since they involve additional assumptions.

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0020-7683/01/$ - see front matter
PII: S0020-7683(01)00012-9

The extension of the asymptotic theory from cylindrical to spherical shells is not trivial. On a cylindrical surface a cartesian coordinate system can be constructed, and a process utilizing differential operators can be applied in a systematic way, since the subsequent differential equations are linear with constant coefficients. No cartesian coordinate system can be constructed on the sphere, and the use of for instance spherical coordinates will lead to differential equations with variable coefficients, not suited for a step-by-step deduction using differential operators.

In this paper we leave the surface coordinates undetermined, and deduce the two-dimensional tensor equations of equilibrium by evaluating the three-dimensional covariant derivatives in terms of two-dimensional tensors. The resulting two-dimensional differential equations are linear with constant coefficients, and one can use the two-dimensional covariant derivative (and the Laplacian operator) as operators in the step-by-step deduction of the final equations. However, since the spherical surface is non-Euclidean, the order of covariant differentiation is important and must be observed.

We obtain as result an asymptotic expansion of the shell equations, which are exact in the sense of having the correct coefficients for all the powers of thickness. As with the asymptotic equations for plates (Niordson, 1979) and cylindrical shells (Niordson, 2000) the first power of the thickness is absent in the asymptotic expansion. To limit the presentation we give details of only the first (membrane theory) and the third order theory (bending theory), but the method is general and will produce any desired order.

It is worthwhile to stress, that the asymptotic equations are not confined to the "interior" of the shell, but hold right up to the boundary. Assuming that the boundary conditions, whether in terms of stresses or displacements can be represented by a power series in z, (the normal coordinate) at each point of the boundary throughout the thickness of the shell, we can satisfy any one of them by using a sufficiently high order theory.

The equations are verified in a single case where an exact solution could be obtained independently of this theory.

2. Equations of motion

Let $x_i = (x, y, z)$ be *normal coordinates* (Niordson, 1985, p. 35), where z is the distance from the middle surface, $z = +h$ being the outer surface, and $z = -h$ the inner surface of the shell. The coordinates x and y on the middle surface are arbitrary. We shall assume that the shell is closed with no boundary, or that the boundary of a (simply connected) element of the shell is defined by the normals to the middle surface along one or two simple non-intersecting closed curves C on this surface.

We shall furthermore assume that the shell performs small harmonic vibrations of amplitude $u^i = (u, v, w)$ and angular frequency ω. The material of the shell is supposed to be homogeneous, isotropic and linearly elastic, following Hooke's law.

In normal coordinates the covariant metric tensor g_{ij} has the components [1]

$$g_{\alpha\beta} = a_{\alpha\beta} - 2d_{\alpha\beta}z + d^{\gamma}_{\alpha}d_{\gamma\beta}z^2, g_{\alpha 3} = g_{3\alpha} = 0, \quad g_{33} = 1,$$

where $a_{\alpha\beta}(x, y)$ and $d_{\alpha\beta}(x, y)$ are the first and second fundamental tensors of the middle surface of the shell.

A spherical shell is characterized by its constant curvature. In mixed form the curvature tensor d^{α}_{β} is plus or minus Kronecker's delta δ^{α}_{β} divided by the radius R of the middle surface. For the sake of getting the formulas as simple as possible, we shall use the radius of middle surface as unit of length, so that $R = 1$.

The sign of the curvature tensor depends on the choice of surface coordinates, and so that there is a minus sign if the normal to this surface points out of the sphere. Taking this to be the case, we have

$$d_{\alpha\beta} = -a_{\alpha\beta}.$$

[1] Latin indices are used for the range 1, 2, 3 and Greek indices for the range 1, 2.

Therefore, in the case of a spherical shell, the metric tensor can be written as

$$g_{\alpha\beta}(x,y,z) = a_{\alpha\beta}(x,y)(1+z)^2, g_{\alpha 3} = g_{3\alpha} = 0, \quad g_{33} = 1.$$

The contravariant components of the metric tensor are given by

$$g^{ij} = \frac{\text{cofactor}(g_{ij})}{\det(g_{ij})},$$

from which it follows that

$$g^{\alpha\beta}(x,y,z) = \frac{a^{\alpha\beta}(x,y)}{(1+z)^2}, g^{\alpha 3} = g^{3\alpha} = 0, \quad g^{33} = 1.$$

Also the Christoffel symbols $\left\{ {i \atop jk} \right\}$ are in general functions of z, however, we find that the lower order symbols $\left\{ {\alpha \atop \beta\gamma} \right\}$ are independent of z.

The remaining Christoffel symbols are

$$\left\{ {3 \atop 33} \right\} = \left\{ {\alpha \atop 33} \right\} = \left\{ {3 \atop \alpha 3} \right\} = \left\{ {3 \atop 3\alpha} \right\} = 0,$$

$$\left\{ {3 \atop \alpha\beta} \right\} = -(1+z)a_{\alpha\beta},$$

$$\left\{ {\alpha \atop 3\beta} \right\} = \frac{1}{1+z}\delta^{\alpha}_{\beta}.$$

Let the displacement vector in normal coordinates be

$$u_i(x,y,z).$$

In the following, whenever convenient, we shall use w to denote u_3.

The stress tensor $\sigma_{ij}(x,y,z)$ is given in terms of the displacements by Hooke's law,

$$\sigma_{ij} = G\left(\mathscr{D}_i u_j + \mathscr{D}_j u_i + \frac{2v}{1-2v} g_{ij}\mathscr{D}^k u_k\right),$$

or, in dimensionless form

$$\Sigma_{ij} = \mathscr{D}_i u_j + \mathscr{D}_j u_i + \frac{2v}{1-2v} g_{ij}\mathscr{D}^k u_k, \tag{1}$$

where G is the shear modulus. The script letter $\mathscr{D}$ denotes the covariant derivative (or contravariant derivative as the case may be) in *three* dimensions, while we shall use the symbol D on the middle surface of the shell, i.e. in two dimensions.

The equations of motion are

$$\mathscr{D}^i \Sigma_{ij} + \Lambda u_j = 0, \tag{2}$$

with

$$\Lambda = \rho\omega^2/G, \tag{3}$$

where ρ is the density of the material and ω the circular frequency.

Using Eqs. (1) and (2) the equations of motions may be expressed in terms of the displacements

$$\mathscr{D}^i\mathscr{D}_iu_j + \mathscr{D}^i\mathscr{D}_ju_i + \frac{2v}{1-2v}\mathscr{D}_j\mathscr{D}^ku_k + \Lambda u_j = 0$$

and since the order of covariant differentiation in the three-dimensional Euclidean space is immaterial, we get

$$\mathscr{D}_i\mathscr{D}^iu_j + \frac{1}{1-2v}\mathscr{D}_j\mathscr{D}^ku_k + \Lambda u_j = 0. \tag{4}$$

The equations of motion are split into the tangential and the normal directions, so that the motion in the tangential plane is governed by

$$\mathscr{D}_i\mathscr{D}^iu_\beta + \frac{1}{1-2v}\mathscr{D}_\beta\mathscr{D}^ku_k + \Lambda u_\beta = 0 \tag{5}$$

and in the normal direction by

$$\mathscr{D}_i\mathscr{D}^iw + \frac{1}{1-2v}\mathscr{D}_3\mathscr{D}^ku_k + \Lambda w = 0. \tag{6}$$

To proceed, we reduce the three-dimensional tensors and covariant derivatives in the normal coordinate system to two-dimensional tensors and covariant derivatives on spherical surfaces of radius $R+z$.

In three dimensions the gradient of the displacement vector u_i is the second order tensor

$$\mathscr{A}_{jk} = \mathscr{D}_ju_k = u_{k,j} - \left\{ {p \atop jk} \right\} u_p,$$

where $\mathscr{D}_j$ is the covariant derivative.

The displacement gradient in two dimensions is

$$A_{\beta\gamma} = D_\beta u_\gamma = u_{\gamma,\beta} - \left\{ {\delta \atop \beta\gamma} \right\} u_\delta,$$

and we find the first components of the gradient to be

$$\mathscr{A}_{\alpha\beta} = A_{\alpha\beta} + a_{\alpha\beta}(1+z)w.$$

The remaining components are

$$\mathscr{A}_{\alpha 3} = u_{3,\alpha} - \left\{ {\gamma \atop \alpha 3} \right\} u_\gamma = w_{,\alpha} - \frac{1}{1+z}u_{\alpha,}$$

$$\mathscr{A}_{3\alpha} = u_{\alpha,3} - \left\{ {\gamma \atop 3\alpha} \right\} u_\gamma = u_{\alpha,3} - \frac{1}{1+z}u_{\alpha,}$$

$$\mathscr{A}_{33} = w_{,3}.$$

The trace of the gradient is an invariant,

$$\mathscr{D}^ku_k = g^{ik}\mathscr{D}_ju_k = g^{ik}\mathscr{A}_{jk} = g^{\alpha\beta}\mathscr{A}_{\alpha\beta} + \mathscr{A}_{33} = \frac{1}{(1+z)^2}(D^\gamma u_\gamma + 2(1+z)w) + w_{,3}.$$

The Laplacian of the first two components of the displacement vector is

$$\mathscr{D}_i\mathscr{D}^iu_\gamma = g^{ij}\mathscr{D}_i\mathscr{A}_{j\gamma} = g^{\alpha\delta}\mathscr{D}_\alpha\mathscr{A}_{\delta\gamma} + \mathscr{D}_3\mathscr{A}_{3\gamma},$$

where the first term is

$$\mathscr{D}_\alpha \mathscr{A}_{\delta\gamma} = \mathscr{A}_{\delta\gamma,\alpha} - \left\{ \begin{matrix} \epsilon \\ \delta\alpha \end{matrix} \right\} (A_{\epsilon\gamma} + a_{\epsilon\gamma}(1+z)w) + (1+z)a_{\delta\alpha}\left(u_{\gamma,3} - \frac{1}{1+z}u_\gamma\right)$$
$$- \left\{ \begin{matrix} \epsilon \\ \gamma\alpha \end{matrix} \right\} (A_{\delta\epsilon} + a_{\delta\epsilon}(1+z)w) + (1+z)a_{\gamma\alpha}\left(w_{,\delta} - \frac{1}{1+z}u_\delta\right),$$

which reduces to

$$D_\alpha A_{\delta\gamma} + (1+z)(a_{\delta\gamma}w_{,\alpha} + a_{\delta\alpha}u_{\gamma,3} + a_{\gamma\alpha}w_{,\delta}) - a_{\delta\alpha}u_\gamma - a_{\gamma\alpha}u_\delta.$$

The second term is

$$\mathscr{D}_3 \mathscr{A}_{3\gamma} = \mathscr{A}_{3\gamma,3} - \left\{ \begin{matrix} q \\ 33 \end{matrix} \right\} \mathscr{A}_{q\gamma} - \left\{ \begin{matrix} q \\ \gamma 3 \end{matrix} \right\} \mathscr{A}_{3q} = \left(u_{\gamma,3} - \frac{1}{1+z}u_\gamma\right)_{,3} - \left\{ \begin{matrix} q \\ \gamma 3 \end{matrix} \right\} \mathscr{A}_{3q}$$
$$= u_{\gamma,33} - \frac{2}{1+z}u_{\gamma,3} + \frac{2}{(1+z)^2}u_\gamma.$$

Hence

$$\mathscr{D}_i \mathscr{D}^i u_\gamma = \frac{1}{(1+z)^2}\Delta u_\gamma + \frac{2}{1+z}w_{,\gamma} - \frac{1}{(1+z)^2}u_\gamma + u_{\gamma,33},$$

where Δ is the two-dimensional Laplacian $D_\gamma D^\gamma$.

The equations of motion in the tangent plane (5) are therefore equivalent to

$$\frac{1}{(1+z)^2}\Delta u_\alpha + \frac{2}{1+z}w_{,\alpha} - \frac{1}{(1+z)^2}u_\alpha + u_{\alpha,33} + \frac{1}{1-2\nu}\left(\frac{1}{(1+z)^2}D_\alpha(\Phi + 2(1+z)w) + D_\alpha w_{,3}\right)$$
$$+ \Lambda u_\alpha$$
$$= 0, \tag{7}$$

where

$$\Phi = D^\gamma u_\gamma \tag{8}$$

is the trace of the two-dimensional displacement gradient.

The first term in the equation of motion (6) in the normal direction is the three-dimensional Laplacian of w

$$\mathscr{D}_i \mathscr{D}^i w = g^{\alpha\delta}\mathscr{D}_\alpha \mathscr{A}_{\delta 3} + \mathscr{D}_3 \mathscr{A}_{33},$$

where

$$\mathscr{D}_\alpha \mathscr{A}_{\delta 3} = \mathscr{A}_{\delta 3,\alpha} - \left\{ \begin{matrix} \gamma \\ \delta\alpha \end{matrix} \right\} \mathscr{A}_{\gamma 3} - \left\{ \begin{matrix} 3 \\ \delta\alpha \end{matrix} \right\} \mathscr{A}_{33} - \left\{ \begin{matrix} \gamma \\ 3\alpha \end{matrix} \right\} \mathscr{A}_{\delta\gamma}$$
$$= w_{,\delta\alpha} - \frac{1}{1+z}u_{\delta,\alpha} - \left\{ \begin{matrix} \gamma \\ \delta\alpha \end{matrix} \right\}\left(w_{,\gamma} - \frac{1}{1+z}u_\gamma\right) - \left\{ \begin{matrix} 3 \\ \delta\alpha \end{matrix} \right\} w_{,3} - \left\{ \begin{matrix} \gamma \\ 3\alpha \end{matrix} \right\}(A_{\delta\gamma} + a_{\delta\gamma}(1+z)w)$$
$$= D_\alpha D_\delta w - \frac{2}{1+z}D_\alpha u_\delta + (1+z)a_{\delta\alpha}w_{,3} + a_{\delta\alpha}w$$

and hence

$$\mathscr{D}_i \mathscr{D}^i w = \frac{1}{(1+z)^2}\left(\Delta w - \frac{2}{1+z}\Phi + 2(1+z)w_{,3} + 2w\right) + w_{,33}.$$

Now

$$\mathscr{D}_3\mathscr{D}^k u_k = \left[\frac{1}{(1+z)^2}(D^\gamma u_\gamma + 2(1+z)w) + w_{,3}\right]_{,3}$$
$$= -\frac{2\Phi}{(1+z)^3} + \frac{\Phi_{,3}}{(1+z)^2} - \frac{2w}{(1+z)^2} + \frac{2w_{,3}}{1+z} + w_{,33},$$

so that the equation of motion (6) in the normal direction is equivalent to

$$\frac{1}{(1+z)^2}\left(\Delta w - \frac{2}{1+z}\Phi + 2(1+z)w_{,3} - 2w\right) + w_{,33}$$
$$+\frac{1}{1-2v}\left(-\frac{2\Phi}{(1+z)^3} + \frac{\Phi_{,3}}{(1+z)^2} - \frac{2w}{(1+z)^2} + \frac{2w_{,3}}{1+z} + w_{,33}\right) + \Lambda w$$
$$= 0. \tag{9}$$

The components of the stress tensor (1) in the tangent plane are found to have the following form in terms of the displacements

$$\Sigma_{\alpha\beta} = D_\beta u_\alpha + D_\alpha u_\beta + 2a_{\alpha\beta}(1+z)w + \frac{2v}{1-2v}a_{\alpha\beta}(\phi + 2(1+z)w + (1+z)^2 w_{,3}). \tag{10}$$

The shear stress vector is

$$\Sigma_{\alpha 3} = \Sigma_{3\alpha} = u_{\alpha,3} + w_{,\alpha} - \frac{2}{1+z}u_{\alpha,} \tag{11}$$

and the normal stress is

$$\Sigma_{33} = 2w_{,3} + \frac{2v}{1-2v}\frac{\phi + 2(1+z)w + w_{,3}}{(1+z)^2}. \tag{12}$$

This rounds up the basic relations of motion and stresses expressed in terms of the displacement vector in three dimensions. We proceed with an asymptotic expansion of all displacements and stresses.

3. Asymptotic expansion

For given conditions at the boundary the displacements will depend on the thickness of the shell, and this dependence will be represented by the following expansion of the covariant components

$$u_i(x,y,z) = \sum_{n=0}^{\infty} u_i^{(n)}(x,y,z)\epsilon^n, \tag{13}$$

where the dimensionless number

$$\epsilon = h/L$$

is assumed to be much smaller than 1. The length $L = \mathrm{O}(R)$ is of the same order as the radius of the middle surface, i.e. 1. We could have taken $L = R$ but since we have taken $R = 1$, some confusion might arise.

The functions $u_i^{(n)}(x,y,z)$ are now expanded in Taylor series at $z = 0$, i.e.

$$u_i(x,y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{i}{m!}U_i^{(n,m)}(x,y)z^m\epsilon^n,$$

or

$$u_i = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{i}{m!} U_i^{(n,m)} z^m \epsilon^n,$$

where

$$U_i^{(n,m)} = \frac{\partial^m u_i^{(n)}}{\partial z^m}(x,y,0), \quad n,m = 0,1,\ldots$$

are the partial derivatives of $u_i^{(n)}$ with respect to z at the middle surface. It follows that the displacements of the middle surface, with which the two-dimensional theory shall deal, are given by

$$u_i(x,y,0) = \sum_{n=0}^{\infty} U_i^{(n,0)} \epsilon^n.$$

From the definition of ϕ it follows that this invariant can be expanded in the following double series

$$\phi(x,y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{i}{m!} \Phi^{(n,m)}(x,y) z^m \epsilon^n,$$

where

$$\Phi^{(n,m)}(x,y) = D^\alpha U_\alpha^{(n,m)}(x,y).$$

The stresses are expanded in the same way as the displacements,

$$\Sigma_{ij}(x,y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{m!} S_{ij}^{(n,m)}(x,y) z^m \epsilon^n. \tag{14}$$

The dependence of Λ and h is given by the asymptotic expansion

$$\Lambda = \sum_{n=0}^{\infty} \Lambda^{(n)} \epsilon^n. \tag{15}$$

Substituting the displacements and Λ into the equations of motion (7) and (9) and equating the coefficients for all powers of z and ϵ to zero, we get the following set of vector equations for equilibrium in the tangent plane,

$$\begin{aligned} &D_\alpha \Phi^{(n,m)} + (\Delta - 1 - m(m-1))U_\alpha^{(n,m)} + 2mU_\alpha^{(n,1+m)} + U_\alpha^{(n,m+2)} + m(m+3)D_\alpha W^{(n,m-1)} \\ &\quad + 2(m+2)D_\alpha W^{(n,m)} + D_\alpha W^{(n,m+1)} + \frac{2\nu}{1-2\nu}[D_\alpha \Phi^{(n,m)} + m(m+1)D_\alpha W^{(n,m-1)} + (m+1)D_\alpha W^{(n,m)} \\ &\quad + D_\alpha W^{(n,m+1)}] + \sum_{r=0}^{n} \Lambda_{(r)}[m(m-1)U_\alpha^{(n-r,m-2)} + 2mU_\alpha^{(n-r,m-1)} + U_\alpha^{(n-r,m)}] \\ &\quad = 0 \end{aligned} \tag{16}$$

and a set of scalar equations for equilibrium in the normal direction,

$$
\begin{aligned}
&(m-4)\Phi^{(n,m)} + \Phi^{(n,m+1)} + m(\Delta - 4 - 2m(m-1))W^{(n,m-1)} + (\Delta - 4 + 2m + 6m^2)W^{(n,m)} \\
&\quad + 2(3m+2)W^{(n,m+1)} + 2W^{(n,m+2)} + \frac{2v}{1-2v}[(m-2)\Phi^{(n,m)} + \Phi^{(n,m+1)} + (m-2)m(m+1)W^{(n,m-1)} \\
&\quad + (m+1)(3m-2)W^{(n,m)} + (3m+2)W^{(n,m+1)} + W^{(n,m+2)}] + \sum_{r=0}^{n} \Lambda_{(r)}[(m-2)(m-1)mW^{(n-r,m-3)} \\
&\quad + 3(m-1)mW^{(n-r,m-2)} + 3mW^{(n-r,m-1)} + W^{(n-r,m)}] \\
&\quad = 0. \qquad (17)
\end{aligned}
$$

Here $\Delta = a^{\alpha\beta}D_\alpha D_\beta$ is the (two-dimensional) Laplacian operator on the middle surface. Eqs. (16) and (17) must hold for all integers $n, m \geqslant 0$.

Substituting the displacements into the expression (10), (11) and (12) for the stress components and equating the coefficients for all powers of ϵ and z to zero, we get the shear stress vector

$$
S_{\alpha 3}^{(n,m)} = -mS_{\alpha 3}^{(n,m-1)} - 2U_\alpha^{(n,m)} + mU_\alpha^{(n,m)} + U_\alpha^{(n,m+1)} + mD_\alpha W^{(n,m-1)} + D_\alpha W^{(n,m)}, \qquad (18)
$$

the normal stress

$$
\begin{aligned}
S_{33}^{(n,m)} &= -m(m-1)S_{33}^{(n,m-2)} - 2mS_{33}^{(n,m-1)} + m(m-1)W^{(n,m-1)} + 2mW^{(n,m)} + W^{(n,m+1)} + \frac{2v}{1-2v} \\
&\quad \times [\Phi^{(n,m)} + mW^{(n,m-1)} + m^2 W^{(n,m-1)} + 2W^{(n,m)} + 2mW^{(n,m)} + W^{(n,m+1)}], \qquad (19)
\end{aligned}
$$

and the in-plane stress tensor

$$
\begin{aligned}
S_{\alpha\beta}^{(n,m)} &= D_\alpha U_\beta^{(n,m)} + D_\beta U_\alpha^{(n,m)} + \frac{a_{\alpha\beta}}{1-2v}[2v\Phi^{(n,m)} + 2m(1 + (m-1)v)W^{(n,m-1)} \\
&\quad + (2 + 4mv)W^{(n,m)} + 2vW^{(n,m+1)}].
\end{aligned}
$$

The boundary conditions at the surfaces $z = \pm h$ are given by

$$
\begin{aligned}
\Sigma_{i3}(x, y, +h) &= P_i^+(x, y), \\
\Sigma_{i3}(x, y, -h) &= P_i^-(x, y),
\end{aligned} \qquad (21)
$$

where P_i^+ and P_i^- are the external loads on the outer and inner surface of the spherical shell respectively. These are six boundary conditions, three for the inner and three for the outer surface.

The static problem of an externally loaded spherical shell requires that we take $\Lambda = 0$ and prescribe the external forces P_i^+ and P_i^- on the outer and inner surface as function of the surface coordinates x and y.

For the dynamic problem of free vibrations, we keep $\Lambda \neq 0$ but take the external forces P_i^+ and P_i^- to be equal to zero.

The further treatment of both cases is quite similar. The static problem leads to a system of three inhomogeneous differential equations with a unique solution. The dynamic one leads to a system of three homogeneous equations, i.e. an eigenvalue problem. In both cases the main effort lies in determining the coefficients of the displacement functions and their determinant.

To avoid a tedious repetition we shall confine our further analysis to the case of free vibrations.

The equations for all derivatives with respect to z of the displacements and stresses are derived. In Section 4 we shall proceed to eliminate the unknown derivatives, expressing them all in terms of the displacements of the middle surface, which yields the two-dimensional shell theory.

4. Elimination of the derivatives

In order to obtain the two-dimensional equations for the shell, we must eliminate all derivatives with respect to z by expressing $U_\alpha^{(n,m)}$, $\Phi^{(n,m)}$ and $W^{(n,m)}$ in terms of the zero order derivatives $U_\alpha^{(r,0)}$, $\Phi^{(r,0)}$ $W^{(r,0)}$ for $r = 0, 1, 2, \dots, n$.

Introducing the expansion of the stresses (14) into the boundary conditions for the outer and inner surface (17) we get

$$\Sigma_{i3}(x, y, \pm h) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{m!}S_{i3}^{(n,m)}(\pm L)^m \epsilon^m \epsilon^n = 0, \quad i = 1, 2, 3,$$

which holds true if

$$\sum_{r=0}^{n}\frac{(\pm L)^{n-r}}{(n-r)!}S_{i3}^{(r,n-r)} = 0, \quad i = 1, 2, 3, \;\; n = 0, 1, 2, \dots$$

or for the condition at the outer surface,

$$\sum_{r=0}^{n-1}\frac{L^{n-r}}{(n-r)!}S_{i3}^{(r,n-r)} + S_{i3}^{(n,0)} = 0, \quad i = 1, 2, 3, \;\; n = 0, 1, 2, \dots \tag{22}$$

For $m = 0$ Eqs. (18) and (19) yield together with Eq. (22) for the first derivatives of the displacements in the tangent plane

$$\begin{aligned} U_\alpha^{(n,1)} &= 2U_\alpha^{(n,0)} - D_\alpha W^{(n,0)} + S_{\alpha 3}^{(n,0)}, \\ U_\alpha^{(n,1)} &= 2U_\alpha^{(n,0)} - D_\alpha W^{(n,0)} - \sum_{r=0}^{n-1}\frac{L^{n-r}}{(n-r)!}S_{\alpha 3}^{(r,n-r)}, \end{aligned} \tag{23}$$

and correspondingly for the first derivatives of the normal displacement

$$\begin{aligned} W^{(n,1)} &= -\frac{v}{1-v}(\Phi^{(n,0)} + 2W^{(n,0)}) + \frac{1-2v}{2(1-v)}S_{33}^{(n,0)}, \\ W^{(n,1)} &= -\frac{v}{1-v}(\Phi^{(n,0)} + 2W^{(n,0)}) + \frac{1-2v}{2(1-v)}\sum_{r=0}^{n-1}\frac{L^{n-r}}{(n-r)!}S_{33}^{(r,n-r)}. \end{aligned} \tag{24}$$

For any value of n all higher order derivatives are determined successively from the two lower order derivatives from the equations of motion, which we solve for the derivatives of order $m+2$. Thus

$$\begin{aligned} U_\alpha^{(n,m+2)} = {} & -D_\alpha\Phi^{(n,m)} + (1 - \Delta + m - m^2)U_\alpha^{(n,m)} - 2mU_\alpha^{(n,m+1)} - 3mD_\alpha W^{(n,m-1)} - m^2 D_\alpha W^{(n,m-1)} \\ & - 4D_\alpha W^{(n,m)} - 2mD_\alpha W^{(n,m)} - D_\alpha W^{(n,m+1)} - \frac{2v}{1-2v}[D_\alpha\Phi^{(n,m)} + m(1+m)D_\alpha W^{(n,m-1)} \\ & + 2(m+1)D_\alpha W^{(n,m)} + D_\alpha W^{(n,m+1)}] - \sum_{r=0}^{n}\Lambda^{(r)}[U_\alpha^{(n-r,m)} + 2mU_\alpha^{(n-r,m-1)} + m(m-1)U_\alpha^{(n-r,m-2)}], \end{aligned} \tag{25}$$

where each term is a covariant vector, and

$$W^{n,m+2} = \frac{v}{1-v}[((2-m)\Phi^{(n,m)} - \Phi^{(n,m+1)} - (m-2)m(m+1)W^{(n,m-1)} - (m+1)(3m-2)W^{(n,m)} - (3m+2)W^{(n,m+1)})] + \frac{1-2v}{2(1-v)}[(4-m)\Phi^{(n,m)} - \Phi^{(n,m+1)} - m(\Delta - 4 - 2m + 2m^2)W^{(n,m-1)} + (4 - \Delta - 2m - 6m^2)W^{(n,m)} - 2(2+3m)W^{(n,m+1)} - \sum_{r=0}^{n} \Lambda^{(r)}((m-2)(m-1)mW^{(n-r,-3+m)} + 3m(m-1)W^{(n-r,m-2)} + 3mW^{(n-r,m-1)} + W^{(n-r,m)})], \tag{26}$$

where each term is invariant.

Taking the contravariant derivative of Eq. (25) we find

$$\Phi^{(n,m+2)} = -\Delta\Phi^{(n,m)} + D^\alpha(1 - \Delta + m - m^2)U_\alpha^{(n,m)} - 2m\Phi^{(n,m+1)} - 3m\Delta W^{(n,m-1)} - m^2\Delta W^{(n,m-1)} - 4\Delta W^{(n,m)} - 2m\Delta W^{(n,m)} - \Delta W^{(n,m+1)} + \frac{2v}{1-2v}[-\Delta\Phi^{(n,m)} - m(1+m)\Delta W^{(n,m-1)} - 2\Delta W^{(n,m)} - 2m\Delta W^{(n,m)} - \Delta W^{(n,m+1)}] + \Lambda[-m(m-1)\Phi^{(n,m-2)} - 2m\Phi^{(n,m-1)} - \Phi^{(n,m)}].$$

The covariant derivatives are not commutative on the sphere. Thus

$$D^\alpha \Delta^n A_\alpha = (\Delta + 1)^n D^\alpha A_\alpha,$$

where n is any positive number and A_α is any covariant vector. Hence, after applying this rule and collecting terms, we get

$$\Phi^{(n,m+2)} = (m(m-1) - 2\Delta)\Phi^{(n,m)} - 2m\Phi^{(n,m+1)} - m(m+3)\Delta W^{(n,m-1)} - 2(m+2)\Delta W^{(n,m)} - \Delta W^{(n,m+1)} - \frac{2v}{1-2v}[\Delta\Phi^{(n,m)} + m(m+1)\Delta W^{(n,m-1)} + 2(m+1)\Delta W^{(n,m)} + \Delta W^{(n,m+1)}] - \Lambda[m(m-1)\Phi^{(n,m-2)} + 2m\Phi^{(n,m-1)} + \Phi^{(n,m)}], \tag{27}$$

where again each term is invariant.

We have now the tools for eliminating the derivatives. For $n = 0$ we get from Eqs. (23) and (24) the first derivatives with respect to z

$$U_\alpha^{(0,1)} = 2U_\alpha^{(0,0)} - D_\alpha W^{(0,0)},$$

$$\Phi^{(0,1)} = 2\Phi^{(0,0)} - \Delta W^{(0,0)},$$

$$W^{(0,1)} = -\frac{v}{1-v}(\Phi^{(0,0)} + 2W^{(0,0)}).$$

The second derivatives are found from Eqs. (25)–(27),

$$U_\alpha^{(0,2)} = (1-\Delta)U_\alpha^{(0,0)} - \frac{D_\alpha(\Phi^{(0,0)} + 2(2-v)W^{(0,0)})}{1-v} - \Lambda^{(0)}U_\alpha^{(0,0)},$$

$$W^{(0,2)} = \frac{\Phi^{(0,0)} + (2+v)(\Delta+2)W^{(0,0)}}{1-v} - \frac{1-2v}{2(1-v)}\Lambda^{(0)}W^{(0,0)},$$

$$\Phi^{(0,2)} = \frac{2-v}{1-v}\Delta(\Phi^{(0,0)} + 2W^{(0,0)}) - \Lambda^{(0)}\Phi^{(0,0)}.$$

Using the same equations, Eqs. (25)–(27), all higher order derivatives can be computed.

For next iteration $n = 1$ we determine the first derivatives

$$U_\alpha^{(1,1)} = 2U_\alpha^{(1,0)} - D_\alpha W^{(1,0)},$$

$$\Phi^{(1,1)} = 2\Phi^{(1,0)} - \Delta W^{(1,0)},$$

$$W^{(1,1)} = -\frac{\nu}{1-\nu}(\Phi^{(1,0)} + 2W^{(1,0)}),$$

etc.

5. The sequence of asymptotic equations

To obtain the equations of motion for each iteration n we apply the condition that the stresses at the inner surface vanish, i.e.

$$\sum_{r=0}^{n} \frac{(-L)^{n-r}}{(n-r)!} S_{i3}^{(r,n-r)} = 0, \quad n = 0, 1, 2, \ldots \tag{28}$$

For $n = 0$ we get

$$-2U_\alpha^{(0,0)} + U_\alpha^{(0,1)} + D_\alpha W^{(0,0)} = 0,$$

$$\nu(\Phi^{(0,0)} + 2W^{(0,0)} - W^{(0,1)}) + W^{(0,1)} = 0,$$

two relations, that are identically satisfied because they were used to derive the first derivatives. However for all $n > 0$ the boundary conditions (28) give non-trivial results.

Thus, for $n = 1$ Eq. (28) yields the lowest order shell equations,

$$(1 + \Delta)U_\alpha^{(0,0)} + \frac{1+\nu}{1-\nu} D_\alpha(\Phi^{(0,0)} + W^{(0,0)}) + \Lambda^{(0)} U_\alpha^{(0,0)} = 0, \tag{29}$$

$$-2\frac{1+\nu}{1-\nu}(\Phi^{(0,0)} + 2W^{(0,0)}) + \Lambda^{(0)} W^{(0,0)} = 0. \tag{30}$$

These equations are the equations of the *membrane state* and precisely the same as those given by the classical theory of spherical shells (Niordson, 1984).

For convenience we shall write them in the form

$$F_1[U_\alpha^{(n,0)}, W^{(n,0)}] + \Lambda^{(0)} U_\alpha^{(n,0)} = 0, \tag{31}$$

$$F_2[U_\alpha^{(n,0)}, W^{(n,0)}] + \Lambda^{(0)} W^{(n,0)} = 0, \tag{32}$$

where F_1 and F_2 are the linear differential operators defined by Eqs. (29) and (30).

For $n = 2$ Eq. (28) is identically satisfied, when $\Lambda^{(0)} U_\alpha^{(0,0)}$ is substituted from Eq. (29) and $\Lambda^{(0)} W^{(0,0)}$ is substituted from Eq. (30). This iteration contributes therefore nothing to the theory just as in the case of plates and cylindrical shells.

For $n = 3$ we find after computing the relevant derivatives the equations of motion

$$F_1[U_\alpha^{(2,0)}, W^{(2,0)}] + L^2 G_1[U_\alpha^{(0,0)}, W^{(0,0)}] + \Lambda^{(2)} U_\alpha^{(0,0)} + \Lambda^{(0)} U_\alpha^{(2,0)} = 0, \tag{33}$$

$$F_2[U_\alpha^{(2,0)}, W^{(2,0)}] + L^2 G_2[U_\alpha^{(0,0)}, W^{(0,0)}] + \Lambda^{(2)} W^{(0,0)} + \Lambda^{(0)} W^{(2,0)} = 0, \tag{34}$$

where the linear operators G_1 and G_2 are found to be

$$G_1[U_\alpha^{(n,0)}, W^{(n,0)}] = \frac{9}{6(1-v)^3}[10(1+\Delta)(1-v)^3 U_\alpha^{(n,0)} + D_\alpha(22 - 26v - (38v^2 + 34v^3 - 4v^2\Delta))\Phi^{(n,0)} + 2D_\alpha(26 - 42v - 26v^2 + 34v^3 + 2\Delta(1 - 4v + v^2))W^{(n,0)}], \tag{35}$$

$$G_3[U_\alpha^{(n,0)}, W^{(n,0)}] = \frac{-1}{3(1-v)^3}[(1 + 18v^2 - 19v^3 + \Delta(3 - 6v + 6v^2))\Phi^{(n,0)} + (\Delta^2(-1+v)^2 - 2 + 26v + 2v^2 - 26v^3 + \Delta(6 - 3v - 3v^2 + 6v^3))W^{(n,0)}]. \tag{36}$$

In the asymptotic expansion we have now reached the lowest order shell theory for bending including terms of order h^2. A higher order theory, including higher powers of h is obtained in a quite similar fashion for values of $n > 3$, but we stop here.

Multiplying Eq. (33) by ϵ^2 and adding it to Eq. (31), we get

$$F_1[u_\alpha, w] + h^2 G_1[u_\alpha, w] + \cdots + \Lambda u_\alpha = 0, \tag{37}$$

$$F_2[u_\alpha, w] + h^2 G_2[u_\alpha, w] + \cdots + \Lambda w = 0, \tag{38}$$

where the dots indicate higher order terms and where the displacements are evaluated at the middle surface. These are the two-dimensional equations for spherical shells in harmonic vibrations. We emphasize that the equations for a spherical shell under static load on the outer and inner surfaces are derived in a very similar manner.

The equations of motion for the case $n = 3$, the so-called bending theory for spherical shells are, when written fully out

$$(1+\Delta)u_\alpha + \frac{1+v}{1-v}D_\alpha(\phi + 2w) - \frac{h^2}{6(1-v)^3}[10(1+\Delta)(1-v)^3 u_\alpha + D_\alpha(22 - 26v - 38v^2 + 34v^3 - 4v^2\Delta)\phi + 2D_\alpha(25 - 38v - 23v^2 + 28v^3 + 2\Delta(1 - 4v + 4v^2))w] + \Lambda u_\alpha = 0, \tag{39}$$

for the displacements in the tangent plane, and

$$-2\frac{1+v}{1-v}(\phi + 2w) - \frac{2h^2}{3(1-v)^3}[(1 + 18v^2 - 19v^3 + \Delta(3 - 6v + 6v^2))\phi + (\Delta^2(-1+v)^2 - 2 + 26v + 2v^2 - 26v^3 + \Delta(6 - v + 3v^2 + 6v^3))w] + \Lambda w = 0, \tag{40}$$

for the normal (radial) displacements.

Writing the displacement vector

$$u_\alpha = D_\alpha \Phi + \epsilon_{\beta\alpha} D^\beta \Psi, \tag{41}$$

where Φ and Ψ are scalar functions of the surface coordinates, and $\epsilon_{\beta\alpha}$ is the alternating tensor, we find, when substituting in Eq. (39) and following the method outlined in Niordson (1984), the following two equations, which replace the vector equation

$$\frac{2}{1-v}((1 - v + \Delta)\Phi + (1+v)w) + \frac{h^2}{6(1-v)^3}[(-20(1-v)^3 + 4\Delta^2 v^2 + \Delta(-32 + 56v + 8v^2 - 24v^3))\Phi - 2(26 - 42v - 26v^2 + 34v^3 + 2\Delta(1 - 4v + v^2))w] + \Lambda\Phi = \kappa, \tag{42}$$

and

$$(\Delta + 2)\Psi + 5\frac{h^2}{3}(\Delta + 2)\Psi + \Lambda\Psi = \mu, \tag{43}$$

where μ and κ are arbitrary harmonic functions.

The third equation of motion, representing equilibrium in normal directions is found to be

$$\begin{aligned} &-2\frac{1+v}{1-v}(\Delta\Phi + 2w) - \frac{2h^2}{3(1-v)^3}[(1 + 18v^2 - 19v^3 + \Delta(3 - 6v + 6v^2))\Delta\Phi + (\Delta^2(1-v)^2 \\ &\quad - 2 + 26v + 2v^2 - 26v^3 + \Delta(6 - 3v - 3v^2 + 6v^3))w] + \Lambda w = 0, \end{aligned} \tag{44}$$

the only difference being that $\Delta\Phi$ replaces ϕ. We note that the only differential operator is the invariant Laplacian operator Δ.

6. Comparison with an exact solution

As a verification we would like to compare our results with exact three-dimensional solutions in at least some special cases. To our knowledge, however, there is only one exact solution for spherical shells available in the literature, the solution for a complete and infinitely thin spherical shell by Lamb (1882).

His solution corresponds to our first iteration of the asymptotic expansion, the membrane theory, and the agreement is complete. But this, of course, also holds for the membrane solutions of the classical shell theories, and there is a need to go further.

Let us therefore consider the case of purely radial vibrations. If $u_\alpha = 0$ and w is independent of the surface coordinates, the equations of motion reduce to one single equation for a constant displacement w, which gives the eigenvalue

$$\Lambda = 4\frac{1+v}{1-v} + \frac{4(1+v)(-1+9v)}{3(1-v)^2}h^2.$$

For purely radial vibrations Lamb gives the frequency equation for the eigenvalue $\hat{h}$

$$f = \frac{\hat{v}\hat{h}a + (\hat{h}^2a^2 - \hat{v})\tan(\hat{h}a)}{(\hat{h}^2a^2 - \hat{v}) - \hat{v}\hat{h}a\tan(\hat{h}a)} - \frac{\hat{v}\hat{h}b + (\hat{h}^2b^2 - \hat{v})\tan(\hat{h}b)}{(\hat{h}^2b^2 - \hat{v}) - \hat{v}\hat{h}b\tan(\hat{h}b)} = 0,$$

where a is the outer and b the inner radius of the shell, and (with our notations)

$$\hat{v} = \frac{2-4v}{1-v}, \quad \hat{h}^2 = \Lambda\frac{1-2v}{2-2v}.$$

Lamb solves the eigenvalue $\hat{h}$ from the frequency equation for an infinitely thin shell. But for a shell of finite thickness, there is no explicit solution of it. To determine $\hat{h}$ for a thin (but not necessarily an infinitely thin) shell, we expand Λ (and hence $\hat{h}^2$) in a power series of the thickness. We introduce for the outer radius $a = 1 + h$ and $b = 1 - h$ and substitute for

$$\Lambda = \sum_{n=0}^{\infty} c_n h^n.$$

Then the frequency equation $f(h) = 0$ becomes an identity for all values of h, and this condition determines the coefficients c_n in the power series for Λ. Taking consecutive derivatives of $f(h)$ with respect to h and evaluating them at $h = 0$, we get

$$\Lambda = 4\frac{1+\nu}{1-\nu} + \frac{4(1+\nu)(-1+9\nu)}{3(1-\nu)^2}h^2 + \cdots$$

in complete agreement with our solution.

It may be interesting to note that for purely radial vibrations of a complete sphere, the second order terms in the thickness are strongly dependent on Poisson's ratio, and vanish for $\nu = 1/9$.

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